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# Quark Mass and Field Anomalous Dimensions to $\mathcal{O}(\alpha_s^5)$

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ABSTRACT:

We present the results of the first complete analytic calculation of the quark mass and field anomalous dimensions to  $\mathcal{O}(\alpha_s^5)$  in QCD.

KEYWORDS: Quantum chromodynamics, Perturbative calculations

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## 1 Introduction

The quark masses depend on a renormalization scale. The dependence is usually referred to as “running” and is governed by the quark mass anomalous dimension,  $\gamma_m$ , defined as:

$$\mu^2 \frac{d}{d\mu^2} m|_{g^0, m^0} = m \gamma_m(a_s) \equiv -m \sum_{i \geq 0} \gamma_i a_s^{i+1}, \quad (1.1)$$

where  $a_s = \alpha_s/\pi = g^2/(4\pi^2)$ ,  $g$  is the renormalized strong coupling constant and  $\mu$  is the normalization scale in the customarily used  $\overline{\text{MS}}$  renormalization scheme. Up to and including four loop level the anomalous dimension is known since long [1–5]. In this paper we will describe the results of calculation of  $\gamma_m$  and a related quantity — the quark field anomalous dimension — in the five-loop order.

The evaluation of the quark mass anomalous dimension with five-loop accuracy has important implications. The Higgs boson decay rate into charm and bottom quarks is proportional to the square of the respective quark mass at the scale of  $m_H$  and the uncertainty from the presently unknown 5-loop terms in the running of the quark mass is of order  $10^{-3}$ . This is comparable to the precision advocated for experiments e.g. at TLEP [6]. Similarly, the issue of Yukawa unification is affected by precise predictions for the anomalous quark mass dimension.

The paper is organized as follows. The next section deals with the overall set-up of the calculations. Then we present our results (Section 3), and a brief discussion (Section 4) as well as a couple of selected applications (Section 5). Our short conclusions are given Section 6.

## 2 Technical preliminaries

To calculate  $\gamma_m$  one needs to find the so-called quark mass renormalization constant,  $Z_m$ , which is defined as the ratio of the bare and renormalized quark masses, viz.

$$Z_m = \frac{m^0}{m} = 1 + \sum_{i,j}^{0 \leq j \leq i} (Z_m)_{ij} \frac{a_s^i}{\epsilon^j}. \quad (2.1)$$

Within the  $\overline{\text{MS}}$  scheme [7, 8] the coefficients  $(Z_m)_{ij}$  are just numbers [9];  $\epsilon \equiv 2 - D/2$  and  $D$  stands for the space-time dimension. Combining eqs. (1.1, 2.1) and using the RG-invariance of  $m^0$ , one arrives at the following formula for  $\gamma_m$ :

$$\gamma_m = \sum_{i \geq 0} (Z_m)_{i1} i a_s^i. \quad (2.2)$$

To find  $Z_m$  one should compute the vector and scalar parts of the quark self-energy  $\Sigma_V(p^2)$  and  $\Sigma_S(p^2)$ . In our convention, the bare quark propagator is proportional to  $[\not{p}(1 + \Sigma_V^0(p^2)) - m_q^0(1 - \Sigma_S^0(p^2))]^{-1}$ . Requiring the finiteness of the renormalized quark propagator and keeping only massless and terms linear in  $m_q$ , one arrives at the following recursive equations to find  $Z_m$

$$Z_m Z_2 = 1 + K_\epsilon \{Z_m Z_2 \Sigma_S^0(p^2)\}, \quad Z_2 = 1 - K_\epsilon \{Z_2 \Sigma_V^0(p^2)\}, \quad (2.3)$$

where  $K_\epsilon \{f(\epsilon)\}$  stands for the singular part of the Laurent expansion of  $f(\epsilon)$  in  $\epsilon$  near  $\epsilon = 0$  and  $Z_2$  is the quark wave function renormalization constant. Eqs. (2.3) express  $Z_m$  through massless propagator-type (that is dependent on one external momentum only) Feynman integrals (FI), denoted as *p-integrals* below.

Eqs. (2.3) require the calculation of a large number<sup>1</sup> the *five-loop* p-integrals to find  $Z_m$  and  $Z_2$  to  $\mathcal{O}(\alpha_s^5)$ .

At present there exists no direct way to analytically evaluate five-loop p-integrals. However, according to (2.1) for a given five-loop p-integral we need to know only its *pole* part in  $\epsilon$  in the limit of  $\epsilon \rightarrow 0$ . A proper use of this fact can significantly simplify our task. The corresponding method—so-called Infrared Rearrangement (IRR)—first suggested in [11] and elaborated further in [12–14] allows to effectively decrease number of loops to be computed by one<sup>2</sup>. In its initial version IRR was not really universal; it was not applicable in some (though rather rare) cases of complicated FI's. The problem was solved by elaborating a special technique of subtraction of IR divergences — the  $R^*$ -operation [15, 16]. This technique succeeds in expressing the UV counterterm of every L-loop Feynman integral in terms of divergent and finite parts of some (L-1)-loop massless propagators.

In our case  $L = 5$  and, using IRR, one arrives at around  $10^5$  four-loop p-integrals. These can, subsequently, be reduced to 28 four-loop masterp-integrals, which are known analytically, including their finite parts, from [17, 18] as well as numerically from [19].

<sup>1</sup> We have used QGRAF [10] to produce around  $10^5$  FI's contributing to the quark self-energy at  $\mathcal{O}(\alpha_s^5)$ .

<sup>2</sup> With the price that resulting one-loop-less p-integrals should be evaluated up to and *including* their constant part in the small  $\epsilon$ -expansion.

We need, thus, to compute around  $10^5$  p-integrals. Their singular parts, in turn, can be algebraically reduced to only 28 master 4-loop p-integrals. The reduction is based on evaluating sufficiently many terms of the  $1/D$  expansion [20] of the corresponding coefficient functions [21].

All our calculations have been performed on a SGI ALTIX 24-node IB-interconnected cluster of eight-cores Xeon computers using parallel MPI-based [22] as well as thread-based [23] versions of FORM [24].

### 3 Results

Our result for the anomalous dimension

$$\gamma_m = - \sum_{i=0}^{\infty} (\gamma_m)_i a_s^{i+1}$$

reads:

$$(\gamma_m)_0 = 1, \quad (\gamma_m)_1 = \frac{1}{16} \left\{ \frac{202}{3} + n_f \left[ -\frac{20}{9} \right] \right\}, \quad (3.1)$$

$$(\gamma_m)_2 = \frac{1}{64} \left\{ 1249 + n_f \left[ -\frac{2216}{27} - \frac{160}{3} \zeta_3 \right] + n_f^2 \left[ -\frac{140}{81} \right] \right\}, \quad (3.2)$$

$$\begin{aligned} (\gamma_m)_3 = & \frac{1}{256} \left\{ \frac{4603055}{162} + \frac{135680}{27} \zeta_3 - 8800 \zeta_5 \right. \\ & + n_f \left[ -\frac{91723}{27} - \frac{34192}{9} \zeta_3 + 880 \zeta_4 + \frac{18400}{9} \zeta_5 \right] \\ & \left. + n_f^2 \left[ \frac{5242}{243} + \frac{800}{9} \zeta_3 - \frac{160}{3} \zeta_4 \right] + n_f^3 \left[ -\frac{332}{243} + \frac{64}{27} \zeta_3 \right] \right\}. \end{aligned} \quad (3.3)$$

$$\begin{aligned} (\gamma_m)_4 = & \frac{1}{4^5} \left\{ \frac{99512327}{162} + \frac{46402466}{243} \zeta_3 + 96800 \zeta_3^2 - \frac{698126}{9} \zeta_4 \right. \\ & - \frac{231757160}{243} \zeta_5 + 242000 \zeta_6 + 412720 \zeta_7 \\ & + n_f \left[ -\frac{150736283}{1458} - \frac{12538016}{81} \zeta_3 - \frac{75680}{9} \zeta_3^2 + \frac{2038742}{27} \zeta_4 \right. \\ & \left. + \frac{49876180}{243} \zeta_5 - \frac{638000}{9} \zeta_6 - \frac{1820000}{27} \zeta_7 \right] \\ & + n_f^2 \left[ \frac{1320742}{729} + \frac{2010824}{243} \zeta_3 + \frac{46400}{27} \zeta_3^2 - \frac{166300}{27} \zeta_4 - \frac{264040}{81} \zeta_5 + \frac{92000}{27} \zeta_6 \right] \\ & \left. + \left[ n_f^3 \left[ \frac{91865}{1458} + \frac{12848}{81} \zeta_3 + \frac{448}{9} \zeta_4 - \frac{5120}{27} \zeta_5 \right] + n_f^4 \left[ -\frac{260}{243} - \frac{320}{243} \zeta_3 + \frac{64}{27} \zeta_4 \right] \right] \right\}. \end{aligned} \quad (3.4)$$

Here  $\zeta$  is the Riemann zeta-function ( $\zeta_3 = 1.202056903 \dots$ ,  $\zeta_4 = \pi^4/90$ ,  $\zeta_5 = 1.036927755 \dots$ ,  $\zeta_6 = 1.017343062 \dots$  and  $\zeta_7 = 1.008349277 \dots$ ). Note that in four-loop order we exactly<sup>3</sup>

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<sup>3</sup>This agreement can be also considered as an important check of all our setup which is completely different from the ones utilized at the four-loop calculations.

reproduce well-known results obtained in [4, 5]. The boxed terms in (3.4) are in full agreement with the results derived previously on the basis of the  $1/n_f$  method in [25–27].

For completeness we present below the result for the quark field anomalous dimension  $\gamma_2 = -\sum_{i=0}^{\infty} (\gamma_2)_i a_s^{i+1}$ :

$$\begin{aligned}
(\gamma_2)_4 = & \frac{1}{4^5} \left\{ \frac{2798900231}{7776} + \frac{17969627}{864} \zeta_3 + \frac{13214911}{648} \zeta_3^2 + \frac{16730765}{864} \zeta_4 - \frac{832567417}{3888} \zeta_5 \right. \\
& + \frac{40109575}{1296} \zeta_6 + \frac{124597529}{1728} \zeta_7 \\
& + n_f \left[ -\frac{861347053}{11664} - \frac{274621439}{11664} \zeta_3 + \frac{1960337}{972} \zeta_3^2 + \frac{465395}{1296} \zeta_4 \right. \\
& \left. + \frac{22169149}{5832} \zeta_5 + \frac{1278475}{1944} \zeta_6 + \frac{3443909}{216} \zeta_7 \right] \\
& + n_f^2 \left[ \frac{37300355}{11664} + \frac{1349831}{486} \zeta_3 - \frac{128}{9} \zeta_3^2 - \frac{27415}{54} \zeta_4 - \frac{12079}{27} \zeta_5 - \frac{800}{9} \zeta_6 - \frac{1323}{2} \zeta_7 \right] \\
& \left. + n_f^3 \left[ -\frac{114049}{8748} - \frac{1396}{81} \zeta_3 + \frac{208}{9} \zeta_4 \right] + n_f^4 \left[ \frac{332}{729} - \frac{64}{81} \zeta_3 \right] \right\}. \tag{3.5}
\end{aligned}$$

The above result is presented for the Feynman gauge; the coefficients  $(\gamma_2)_i$  with  $i \leq 3$  can be found in [28] (for the case of a general covariant gauge and  $SU(N)$  gauge group).

## 4 Discussion

In numerical form  $\gamma_m$  reads

$$\begin{aligned}
\gamma_m = & -a_s - a_s^2 (4.20833 - 0.138889 n_f) \\
& - a_s^3 (19.5156 - 2.28412 n_f - 0.0270062 n_f^2) \\
& - a_s^4 (98.9434 - 19.1075 n_f + 0.276163 n_f^2 + 0.00579322 n_f^3) \\
& - a_s^5 (559.7069 - 143.6864 n_f + 7.4824 n_f^2 + 0.1083 n_f^3 - 0.000085359 n_f^4) \tag{4.1}
\end{aligned}$$

and

$$\begin{aligned}
\gamma_m \Big|_{n_f=3} &= a_s - 3.79167 a_s^2 - 12.4202 a_s^3 - 44.2629 a_s^4 - 198.907 a_s^5, \\
g_m \Big|_{n_f=4} &= a_s - 3.65278 a_s^2 - 9.94704 a_s^3 - 27.3029 a_s^4 - 111.59 a_s^5, \\
g_m \Big|_{n_f=5} &= a_s - 3.51389 a_s^2 - 7.41986 a_s^3 - 11.0343 a_s^4 - 41.8205 a_s^5, \\
\gamma_m \Big|_{n_f=6} &= a_s - 3.37500 a_s^2 - 4.83867 a_s^3 + 4.50817 a_s^4 + 9.76016 a_s^5. \tag{4.2}
\end{aligned}$$

Note that significant cancellations between  $n_f^0$  and  $n_f^1$  terms for the values of  $n_f$  around 3 or so persist also at five-loop order. As a result we observe a moderate growth of the series in  $a_s$  appearing in the quark mass anomalous dimension at various values of active quark flavours (recall that even for scales as small as 2 GeV  $a_s \equiv \frac{\alpha_s}{\pi} \approx 0.1$ ).

$n_f$	3	4	5	6
$(\gamma_m)_4^{\text{exact}}$	198.899	111.579	41.807	-9.777
$(\gamma_m)_4^{\text{APAP}}$ [29]	162.0	67.1	-13.7	-80.0
$(\gamma_m)_4^{\text{APAP}}$ [30]	163.0	75.2	12.6	12.2
$(\gamma_m)_4^{\text{APAP}}$ [31]	164.0	71.6	-4.8	-64.6

**Table 1:** The exact results for  $(\gamma_m)_4$  together with the predictions made with the help of the original APAP method and its two somewhat modified versions.

Similar behavior shows up for  $\gamma_2$ :

$$\begin{aligned}
\gamma_2 = & -0.33333a_s - a_s^2(-1.9583 + 0.08333n_f) \\
& - a_s^3(-10.3370 + 1.0877n_f - 0.01157n_f^2) \\
& - a_s^4(-53.0220 + 10.1090n_f - 0.27703n_f^2 - 0.0023n_f^3) \\
& - a_s^5(-310.0700 + 76.3260n_f - 4.6339n_f^2 + 0.0085n_f^3 + 0.00048n_f^4) \quad (4.3)
\end{aligned}$$

and

$$\begin{aligned}
\gamma_2 \Big|_{n_f=3} &= -0.33333a_s - 1.7083a_s^2 - 7.1779a_s^3 - 25.2480a_s^4 - 122.5300a_s^5, \\
g_2 \Big|_{n_f=4} &= -0.33333a_s - 1.6250a_s^2 - 6.1712a_s^3 - 17.1610a_s^4 - 78.2430a_s^5, \\
g_2 \Big|_{n_f=5} &= -0.33333a_s - 1.5417a_s^2 - 5.1877a_s^3 - 9.6824a_s^4 - 42.9240a_s^5, \\
\gamma_2 \Big|_{n_f=6} &= -0.33333a_s - 1.4583a_s^2 - 4.2274a_s^3 - 2.8251a_s^4 - 16.4710a_s^5. \quad (4.4)
\end{aligned}$$

It is instructive to compare our numerical result for  $(\gamma_m)_4$

$$(\gamma_m)_4 = 559.71 - 143.6n_f + 7.4824n_f^2 + 0.1083n_f^3 - 0.00008535n_f^4 \quad (4.5)$$

with a 15 years old prediction based on the “Asymptotic Páde Approximants” (APAP) method [29] (the boxed term below was used as the input)

$$(\gamma_m)_4^{\text{APAP}} = 530 - 143n_f + 6.67n_f^2 + 0.037n_f^3 - \boxed{0.00008535n_f^4} \quad (4.6)$$

Unfortunately, this impressively good agreement does *not* survive for fixed values of  $n_f$  due to severe cancellations between different powers of  $n_f$  as one can see from the Table 1.

The solution of eq. (1.1) reads:

$$\frac{m(\mu)}{m(\mu_0)} = \frac{c(a_s(\mu))}{c(a_s(\mu_0))}, \quad c(x) = \exp \left\{ \int dx' \frac{\gamma_m(x')}{\beta(x')} \right\}, \quad (4.7)$$

$$\begin{aligned}
c(x) = & (x)^{\tilde{\gamma}_0} \left\{ 1 + d_1x + (d_1^2/2 + d_2)x^2 + (d_1^3/6 + d_1d_2 + d_3)x^3 \right. \\
& \left. + (d_1^4/24 + d_1^2d_2/2 + d_2^2/2 + d_1d_3 + d_4)x^4 + \mathcal{O}(x^5) \right\}, \quad (4.8)
\end{aligned}$$

$$d_1 = -\bar{\beta}_1 \bar{\gamma}_0 + \bar{\gamma}_1, \quad (4.9)$$

$$d_2 = \bar{\beta}_1^2 \bar{\gamma}_0/2 - \bar{\beta}_2 \bar{\gamma}_0/2 - \bar{\beta}_1 \bar{\gamma}_1/2 + \bar{\gamma}_2/2, \quad (4.10)$$

$$d_3 = -\bar{\beta}_1^3 \bar{\gamma}_0/3 + 2 \bar{\beta}_1 \bar{\beta}_2 \bar{\gamma}_0/3 - \bar{\beta}_3 \bar{\gamma}_0/3 + \bar{\beta}_1^2 \bar{\gamma}_1/3 - \bar{\beta}_2 \bar{\gamma}_1/3 - \bar{\beta}_1 \bar{\gamma}_2/3 + \bar{\gamma}_3/3, \quad (4.11)$$

$$d_4 = \bar{\beta}_1^4 \bar{\gamma}_0/4 - 3 \bar{\beta}_1^2 \bar{\beta}_2 \bar{\gamma}_0/4 + \bar{\beta}_2^2 \bar{\gamma}_0/4 + \bar{\beta}_1 \bar{\beta}_3 \bar{\gamma}_0/2 - \bar{\beta}_4 \bar{\gamma}_0/4 - \bar{\beta}_1^3 \bar{\gamma}_1/4 \\ + \bar{\beta}_1 \bar{\beta}_2 \bar{\gamma}_1/2 - \bar{\beta}_3 \bar{\gamma}_1/4 + \bar{\beta}_1^2 \bar{\gamma}_2/4 - \bar{\beta}_2 \bar{\gamma}_2/4 - \bar{\beta}_1 \bar{\gamma}_3/4 + \bar{\gamma}_4/4. \quad (4.12)$$

Here  $\bar{\gamma}_i = (\gamma_m)_i/\beta_0$ ,  $\bar{\beta}_i = \beta_i/\beta_0$  and

$$\beta(a_s) = - \sum_{i \geq 0} \beta_i a_s^{i+2} = -\beta_0 \left\{ \sum_{i \geq 0} \bar{\beta}_i a_s^{i+2} \right\}$$

is the QCD  $\beta$ -function. Unfortunately, the coefficient  $d_4$  in eq. (4.12) does depend on the yet unknown *five-loop* coefficient  $\beta_4$  (up to four loops the  $\beta$ -function is known from [14, 32–39]).

Numerically, the  $c$ -function reads:

$$c(x) \stackrel{=}{=}_{n_f=3} x^{4/9} c_s(x), \quad c(x) \stackrel{=}{=}_{n_f=4} x^{12/25} c_c(x), \quad c(x) \stackrel{=}{=}_{n_f=5} x^{12/23} c_b(x), \quad c(x) \stackrel{=}{=}_{n_f=6} x^{4/7} c_t(x),$$

with

$$c_s(x) = 1 + 0.8950 x + 1.3714 x^2 + 1.9517 x^3 + (15.6982 - 0.11111 \bar{\beta}_4) x^4, \\ c_c(x) = 1 + 1.0141 x + 1.3892 x^2 + 1.0905 x^3 + (9.1104 - 0.12000 \bar{\beta}_4) x^4, \\ c_b(x) = 1 + 1.1755 x + 1.5007 x^2 + 0.17248 x^3 + (2.69277 - 0.13046 \bar{\beta}_4) x^4, \\ c_t(x) = 1 + 1.3980 x + 1.7935 x^2 - 0.68343 x^3 + (-3.5130 - 0.14286 \bar{\beta}_4) x^4. \quad (4.13)$$

## 5 Applications

### 5.1 RGI mass

Eq. (4.7) naturally leads to an important concept: the RGI mass

$$m^{\text{RGI}} \equiv m(\mu_0)/c(a_s(\mu_0)), \quad (5.1)$$

which is often used in the context of lattice calculations. The mass is  $\mu$  and *scheme* independent; in *any* (mass-independent) scheme

$$\lim_{\mu \rightarrow \infty} a_s(\mu)^{-\bar{\gamma}_0} m(\mu) = m^{\text{RGI}}.$$

The function  $c_s(x)$  is used, e.g. by the **ALPHA** lattice collaboration to find the  $\overline{\text{MS}}$  mass of the strange quark at a lower scale, say,  $m_s(2 \text{ GeV})$  from the  $m_s^{\text{RGI}}$  mass determined from lattice simulations (see, e.g. [40]). For example, setting  $a_s(\mu = 2 \text{ GeV}) = \frac{\alpha_s(\mu)}{\pi} = 0.1$ , we arrive at ( $h$  counts loops):

$$m_s(2 \text{ GeV}) = m_s^{\text{RGI}} (a_s(2 \text{ GeV}))^{\frac{4}{9}} \left( 1 + 0.0895 h^2 + 0.0137 h^3 + 0.00195 h^4 \right. \\ \left. + (0.00157 - 0.000011 \bar{\beta}_4) h^5 \right) \quad (5.2)$$

In order to have an idea of effects due the five-loop term in (5.2) one should make a guess about  $\bar{\beta}_4$ . By inspecting lower orders in

$$\beta(n_f = 3) = -\left(\frac{4}{9}\right) \left(a_s + 1.777 a_s^2 + 4.4711 a_s^3 + 20.990 a_s^4 + \bar{\beta}_4 a_s^5\right)$$

one can assume a natural estimate of  $\bar{\beta}_4$  as laying in the interval  $50 - 100$ . With this choice we conclude that the (apparent) convergence of the above series is quite good even at a rather small energy scale of 2 GeV.

On the other hand, the authors of [30] estimate  $\bar{\beta}_4$  in the  $n_f = 3$  QCD as large as -850! With such a huge and negative value of  $\bar{\beta}_4$  the five loop term in (5.2) would amount to 0.01092 and, thus, would significantly exceed the four-loop contribution (0.00195).

## 5.2 Higgs decay into quarks

The decay width of the Higgs boson into a pair of quarks can be written in the form

$$\Gamma(H \rightarrow \bar{f}f) = \frac{G_F M_H}{4\sqrt{2}\pi} m_f^2(\mu) R^S(s = M_H^2, \mu) \quad (5.3)$$

where  $\mu$  is the normalization scale and  $R^S$  is the spectral density of the scalar correlator, known to  $\alpha_s^4$  from [41]

$$\begin{aligned} R^S(s = M_H^2, \mu = M_H) &= 1 + 5.667 a_s + 29.147 a_s^2 + 41.758 a_s^3 - 825.7 a_s^4 \\ &= 1 + 0.2041 + 0.0379 + 0.0020 - 0.00140 \end{aligned} \quad (5.4)$$

where we set  $a_s = \alpha_s/\pi = 0.0360$  (for the Higgs mass value  $M_H = 125$  GeV and  $\alpha_s(M_Z) = 0.118$ ).

Expression (5.3) depends on two phenomenological parameters, namely,  $\alpha_s(M_H)$  and the quark running mass  $m_q$ . In what follows we consider, for definiteness, the dominant decay mode  $H \rightarrow \bar{b}b$ . To avoid the appearance of large logarithms of the type  $\ln \mu^2/M_H^2$  the parameter  $\mu$  is customarily chosen to be around  $M_H$ . However, the starting value of  $m_b$  is usually determined at a much smaller scale (typically around 5-10 GeV [42]). The evolution of  $m_b(\mu)$  from a lower scale to  $\mu = M_h$  is described by a corresponding RG equation which is completely fixed by the quark mass anomalous dimension  $\gamma(\alpha_s)$  and the QCD beta function  $\beta(\alpha_s)$  (for QCD with  $n_f = 5$ ). In order to match the  $\mathcal{O}(\alpha_s^4)$  accuracy of (5.4) one should know *both* RG functions  $\beta$  and  $\gamma_m$  in the five-loop approximation. Let us proceed, assuming conservatively that  $0 \leq \bar{\beta}_4^{n_f=5} \leq 200$ .

The value of  $m_b(\mu = M_H)$  is to be obtained with RG running from  $m_b(\mu = 10 \text{ GeV})$  and, thus, depends on  $\beta$  and  $\gamma_m$ . Using the Mathematica package RunDec<sup>4</sup> [43] and eq. (4.13) we find for the shift from the five-loop term

$$\frac{\delta m_b^2(M_H)}{m_b^2(M_H)} = -1.3 \cdot 10^{-4}(\bar{\beta}_4 = 0) - 4.3 \cdot 10^{-4}(\bar{\beta}_4 = 100) - 7.3 \cdot 10^{-4}(\bar{\beta}_4 = 200)$$

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<sup>4</sup>We have extended the package by including the five-loop effects to the running of  $\alpha_s$  and quark masses.



If we set  $\mu = M_H$ , then the combined effect of  $\mathcal{O}(\alpha_s^4)$  terms as coming from the five-loop running and four-loop contribution to  $R^S$  on

$$\Gamma(H \rightarrow \bar{b}b) = \frac{G_F M_H}{4\sqrt{2}\pi} m_f^2(M_H) R^S(s = M_H^2, M_H) \quad (5.5)$$

is around -2‰ (for  $\bar{\beta}_4 = 100$ ). This should be contrasted to the parametric uncertainties coming from the input parameters  $\alpha_s(M_Z) = 0.1185(6)$  [44] and  $m_b(m_b) = 4.169(8)$  GeV [45] which correspond to  $\pm 1\%$  and  $\pm 4\%$  respectively.

We conclude, that the  $\mathcal{O}(\alpha_s^4)$  terms in (5.4), (5.5) are of no phenomenological relevancy at present. But, the situation could be different if the project of TLEP [6] is implemented. For instance, the uncertainty in  $\alpha_s(M_Z)$  could be reduced to  $\pm 2\%$  and Higgs boson branching ratios with precisions in the permille range are advertised.

## 6 Conclusions

We have analytically computed the anomalous dimensions of the quark mass  $\gamma_m$  and field  $\gamma_2$  in the five loop approximation. The self-consistent description of the quark mass evolution at five loop requires the knowledge of the QCD  $\beta$ -function to the same number of loops. The corresponding, significantly more complicated calculation is under consideration.

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